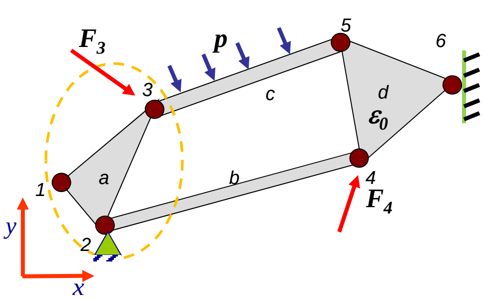
Discrete Systems

The Finite Element Method

Discrete system: definition

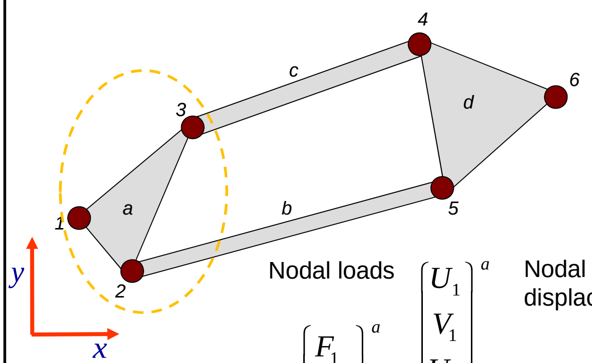
A discrete structure is a mechanical system where parts are connected by discrete nodal points.
Boundary conditions, as well as concentrated loads, are applied on nodes. Possible distributed loads (surface, mass loads) are applied on elements, also.

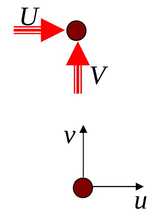
This class of problems can be solved by matrix structural analysis, both for isostatic and hyperstatic configurations.
In particular, it is possible to find the equilibrium configuration (displacement field), reactions (forces and moments on constraints), and stress and strain state in the single parts..
All quantities can be expressed as a function of **nodal displacements**.



The Finite Element Method

Discrete system: 2d example





Nodal loads

$$\{F\}^a = \begin{Bmatrix} F_1 \\ F_2 \\ F_3 \end{Bmatrix}^a = \begin{Bmatrix} U_1 \\ V_1 \\ U_2 \\ V_2 \\ U_3 \\ V_3 \end{Bmatrix}^a$$

Nodal displacements

$$\{d\}^a = \begin{Bmatrix} d_1 \\ d_2 \\ d_3 \end{Bmatrix}^a = \begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \end{Bmatrix}^a$$

$$\{F_1\} = \begin{Bmatrix} U_1 \\ V_1 \end{Bmatrix}$$

$$\{d_1\} = \begin{Bmatrix} u_1 \\ u_1 \end{Bmatrix}$$

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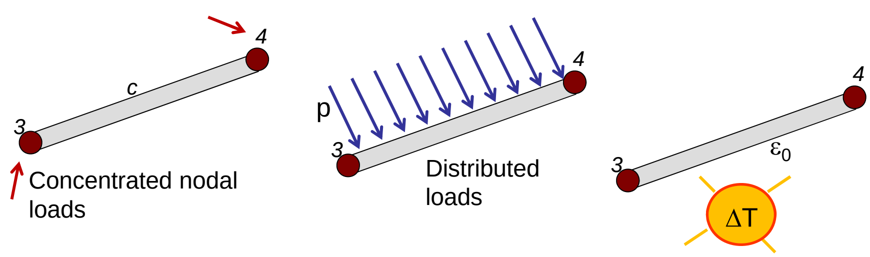
Discrete system: 2d example

Loads acting on a single part



Initial element strain, due to temperature change

Element equilibrium relation:

$$\{F\}^a = [K]^a \{d\}^a + \{F\}_p^a + \{F\}_{\varepsilon 0}^a$$

The Finite Element Method

Discrete system: single element, general case

Element equilibrium relation:

$$\{F\}^a = [K]^a \{d\}^a + \{F\}^a_p + \{F\}^a_{\varepsilon 0}$$

$\{F\}^a$ Nodal load vector

$\{F\}^a_p$ Nodal load vector needed to balance distributed loads

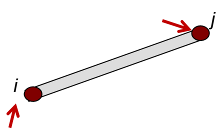
$\{F\}^a_{\varepsilon_0}$ Nodal load vector needed to balance initial strains, preventing node displacements

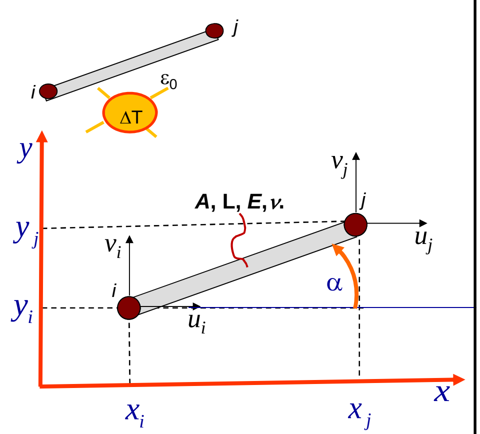
$\{F\}^a_d = [K]^a \{d\}^a$ Nodal load vector which gives rise to the elastic nodal displacements: $\{d\}^a$

The Finite Element Method

Discrete system: 2d link element example

2d link element, of uniform cross section *A*, and length *L*. Elastic material: *E*, *ν*. Reacts to tension-compression, due to concentrated loads at nodes. Temperature variations possible within the element.





Link length

$$L = \sqrt{(x_j - x_i)^2 + (y_j - y_i)^2}$$

Angolo formato dall'asta con l'asse delle ascisse

$$\alpha = \arctan\left(\frac{y_j - y_i}{x_j - x_i}\right)$$

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Discrete system: 2d link element example

Equilibrium relation:

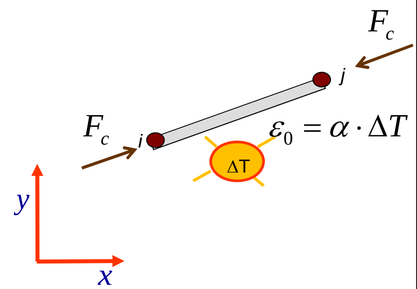
$$\{F\}^a = [K]^a \{d\}^a + \{F\}^a_{\varepsilon 0}$$

Nodal forces needed to prevent deformation due to temperature variations

$$\varepsilon_0 = \alpha \Delta T \qquad \sigma_0 = E\varepsilon_0$$

$$F_c = E \cdot \alpha \cdot \Delta T \cdot A$$

$$\{F\}^a_{\varepsilon_0} = \begin{Bmatrix} F_i \\ F_j \end{Bmatrix}^a_{\varepsilon_0} = E \cdot \alpha \cdot \Delta T \cdot A \cdot \begin{Bmatrix} \cos a \\ \sin a \\ -\cos a \\ -\sin a \end{Bmatrix}$$



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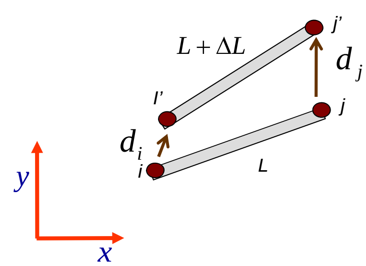
Discrete system: 2d link element example

Equilibrium relation:

$$\{F\}^a = [K]^a \{d\}^a + \{F\}^a_{\varepsilon 0}$$

Nodal displacements

$$\{d\}^a = \begin{Bmatrix} d_i \\ d_j \end{Bmatrix}^a = \begin{Bmatrix} u_i \\ v_i \\ u_j \\ v_j \end{Bmatrix}$$



Link elongation

$$\Delta L = (u_j - u_i)\cos a + (v_j - v_i)\sin a$$

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Discrete system: 2d link element example

Equilibrium relation:

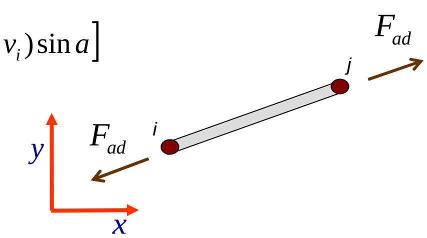
$$\{F\}^a = [K]^a \{d\}^a + \{F\}^a_{\varepsilon 0}$$

Axial force causes a link elongation ΔL

$$F_{ad} = A \cdot E \cdot \frac{\Delta L}{L} = \frac{EA}{L}\left[(u_j - u_i)\cos a + (v_j - v_i)\sin a\right]$$

In a vector form:

$$\{F\}^a_d = \begin{Bmatrix} F_{d_i} \\ F_{d_j} \end{Bmatrix}^a_d = \begin{Bmatrix} -cF_{ad} \\ -sF_{ad} \\ cF_{ad} \\ sF_{ad} \end{Bmatrix}^a_d = \begin{Bmatrix} U_i \\ V_i \\ U_j \\ V_j \end{Bmatrix}$$



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Discrete system: 2d link element example

Equilibrium relation:

$$\{F\}^a = [K]^a \{d\}^a + \{F\}^a_{\varepsilon 0}$$

$$U_i = \frac{EA}{L}\left(+u_i c^2 + v_i sc - u_j c^2 - v_j sc\right)$$

$$V_i = \frac{EA}{L}\left(+u_i sc + v_i s^2 - u_j sc - v_j s^2\right)$$

$$U_j = \frac{EA}{L}\left(-u_i c^2 - v_i sc + u_j c^2 + v_j sc\right)$$

$$V_i = \frac{EA}{L}\left(-u_i sc - v_i s^2 + u_j sc + v_j s^2\right)$$

$c = \cos a$

$s = \sin a$

The Finite Element Method

Discrete system: 2d link element example

Equilibrium relation:

$$\{F\}^a = [K]^a\{d\}^a + \{F\}^a_{\varepsilon 0}$$

Previous expressions may be rearranged in a matrix form:

$$\{F\}^a_d = \frac{EA}{L}\begin{bmatrix} c^2 & sc & -c^2 & -sc \\ sc & s^2 & -sc & -s^2 \\ -c^2 & -sc & c^2 & sc \\ -sc & -s^2 & sc & s^2 \end{bmatrix}^a \begin{Bmatrix} u_i \\ v_i \\ u_j \\ v_j \end{Bmatrix} = \begin{bmatrix} K_{ii} & K_{ij} \\ K_{ji} & K_{jj} \end{bmatrix}^a \begin{Bmatrix} d_i \\ d_j \end{Bmatrix} = [K]^a\{d\}^a$$

$$K_{ij} = \frac{EA}{L}(-1)^{i+j}\begin{bmatrix} c^2 & sc \\ sc & s^2 \end{bmatrix}$$

[*K*] is symmetric, due to energy conservation principle

The Finite Element Method

Discrete system: 2d link element example

Equilibrium relation:

$$\{F\}^a = [K]^a\{d\}^a + \{F\}^a_p + \{F\}^a_{\varepsilon 0}$$

In matrix form:

$$\{F\}^a = \frac{EA}{L}\begin{bmatrix} c^2 & sc & -c^2 & -sc \\ sc & s^2 & -sc & -s^2 \\ -c^2 & -sc & c^2 & sc \\ -sc & -s^2 & sc & s^2 \end{bmatrix}^a \{d\}^a + E\alpha\Delta T\begin{Bmatrix} c \\ s \\ -c \\ -s \end{Bmatrix}$$

State of stress and strain:

$$\{\varepsilon\} = \frac{1}{L}[-c \; -s \; c \; s]\{d\}^a - \alpha\Delta T$$

$$\{\sigma\} = \frac{E}{L}[-c \; -s \; c \; s]\{d\}^a - E\alpha\Delta T$$

The Finite Element Method

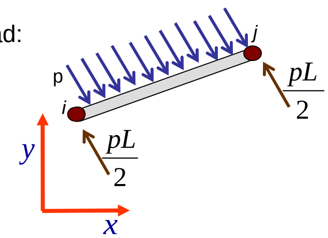
Discrete system: 2d link element example

Equilibrium relation modification, «enhanced» link with distributed loads:

$$\{F\}^a = [K]^a \{d\}^a + \{F\}^a_p + \{F\}^a_{\varepsilon 0}$$

Nodal forces which balance the distributed load:

$$\{F\}^a_p = \begin{Bmatrix} F_i \\ F_j \end{Bmatrix}^a_p = \frac{pL}{2} \begin{Bmatrix} -\sin a \\ \cos a \\ -\sin a \\ \cos a \end{Bmatrix}$$



The Finite Element Method

Discrete system: single element, general case

Element equilibrium relation:

$$\{F\}^a = [K]^a \{d\}^a + \{F\}^a_p + \{F\}^a_{\varepsilon 0}$$

m = number of nodes in the element
l = number of degrees of freedom of a single node

$$\{F\}^a = \begin{Bmatrix} F_1 \\ .. \\ F_i \\ .. \\ F_m \end{Bmatrix}^a \qquad \{d\}^a = \begin{Bmatrix} d_1 \\ .. \\ d_i \\ .. \\ d_m \end{Bmatrix}^a \qquad \{F_i\} = \begin{Bmatrix} F_{i1} \\ .. \\ F_{il} \end{Bmatrix} \qquad \{d_i\} = \begin{Bmatrix} d_{i1} \\ .. \\ d_{il} \end{Bmatrix}$$

load and displacement vectors: *m* x *l* elements

Generalized load and displacement vector components: they can be moments and rotations also.

The Finite Element Method

Discrete system: single element, general case

Element equilibrium relation:

$$\{F\}^a = [K]^a \{d\}^a + \{F\}_p^a + \{F\}_{\varepsilon 0}^a$$

l = degrees of freedom of a single node

$$[K]^a = \begin{bmatrix} K_{11} & .. & K_{1i} & .. & K_{1m} \\ .. & & .. & & .. \\ K_{i1} & .. & K_{ij} & .. & K_{im} \\ .. & & .. & & .. \\ K_{m1} & .. & K_{mj} & .. & K_{mm} \end{bmatrix}$$

$$[K_{ij}] = \begin{bmatrix} K_{ij_{11}} & .. & K_{ij_{1l}} \\ .. & & .. \\ K_{ij_{1l}} & .. & K_{ij_{ll}} \end{bmatrix}$$

$[K]^a$ Element stiffness matrix, *ml* x *ml elements*

The Finite Element Method

Discrete system: structure assembly and resolution

Discrete system static solution: equilibrium imposed on each node of the structure. This leads to a system of equations (linear for the elastic problem), with nodal displacements as unknowns. Solved it, also reactions, state of stress and state of strain can be retrieved. Solution is available at nodes, and in any point of the elements.

Structure external load vector and displacement vector

$$\{R\} = \begin{Bmatrix} R_1 \\ .. \\ R_i \\ .. \\ R_n \end{Bmatrix} \qquad \{R_i\} = \begin{Bmatrix} R_{i1} \\ .. \\ R_{il} \end{Bmatrix} \qquad \{d\} = \begin{Bmatrix} d_1 \\ .. \\ d_i \\ .. \\ d_n \end{Bmatrix}$$

n = number of nodes in the structure

l = number of degrees of freedom of a single node

The Finite Element Method

Discrete system: structure assembly and resolution

Equilibrium condition, *i*-th node of the structure: Sum of external forces and of forces coming from adjacent elements must be zero.

$$\{R_i\} = \sum_a \{F_i\}^a \qquad \text{equivalently:} \quad \left(-\sum_a \{F_i\}^a + \{R_i\} = 0 \right)$$

All elements should be included, not only the neighbouring ones, (non concurring elements give a null contribute).

Element stiffness matrix and element vectors should be expanded to structure dimension for computations.

$$[K]^a = \begin{bmatrix} K_{11} & .. & K_{1n} \\ .. & .. & .. \\ K_{n1} & .. & K_{nn} \end{bmatrix} \quad ([nl\ x\ nl]) \quad \{F\}_p^a = \begin{Bmatrix} F_1 \\ .. \\ F_i \\ .. \\ F_n \end{Bmatrix}_p^a \quad \{F\}_{\varepsilon 0}^a = \begin{Bmatrix} F_1 \\ .. \\ F_i \\ .. \\ F_n \end{Bmatrix}_{\varepsilon 0}^a$$

The Finite Element Method

Discrete system: structure assembly and resolution

Equilibrium condition, *i*-th node

$$\{R_i\} = \sum_a \left(\sum_{j=1}^{n} [K_{ij}]^a \{d_j\} \right)^a + \sum_a \{F_i\}_p^a + \sum_a \{F_i\}_{\varepsilon_0}^a = \sum_{j=1}^{n} \left(\sum_a [K_{ij}]^a \right) \{d_j\} + \sum_a \{F_i\}_p^a + \sum_a \{F_i\}_{\varepsilon_0}^a$$

Equilibrium condition of all nodes of the structure, matrix notation:

$$\{R\} = [K]\{d\} + \{F\}_p + \{F\}_{\varepsilon_0}$$

[K] structure stiffness matrix

$$n.b. \quad [K_{ij}]^a \neq 0 \Leftrightarrow i, j \in a$$

$$[K_{ij}] = \sum_a [K_{ij}]^a$$

$$\{F_i\}_p = \sum_a \{F_i\}_p^a$$

$$\{F_i\}_{\varepsilon_0} = \sum_a \{F_i\}_{\varepsilon_0}^a$$

expanded to structure dimension

The Finite Element Method

Discrete system: resolution

System of *nxl* linear equations, with *nxl* unknown nodal displacements

$$[K]\{d\}=\{R\}-\{F\}_p-\{F\}_{\varepsilon_0}$$

To be solved by sparse matrix linear algebra algorithms

Note:

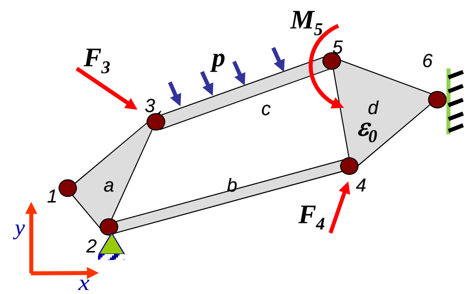
- Equilibrium cannot be found unless enough constraints are imposed, that is, the above system is underdetermined. Analitically, this means that the stiffness matrix *[K]* is singular.

- Boundary conditions impose zero (or fixed) displacements on selected nodes. Reactions arise as a consequence, relative to the constrained degrees of freedom. These can be determined once the unknown displacements have been computed.

The Finite Element Method

Discrete system: resolution example

2d example: solution steps



1: Element connectivity identification.

Elements	Connecting nodes
a	1,2,3
b	2,4
c	3,5
d	4,5,6

2: Identification of the elastic properties of every element: E , n.

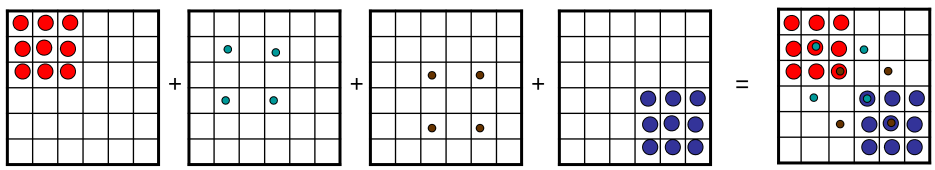
The Finite Element Method

Discrete system: resolution example

3: Computation of element stiffness matrices $[K]^a$ and of concentrated loads equivalent to distributed loads and to effects of initial strain $\{F\}_p^a$, $\{F\}_{\varepsilon_0}^a$, for each element: this allows to calculate the element equilibrium relation: $\{F\}^a = [K]^a \{d\}^a + \{F\}_p^a + \{F\}_{\varepsilon 0}^a$

4: Use of transformation matrices to refer all quantities to a single global reference system.

5: Expansion of element stiffness matrices to structure dimension, prior to their assembly into the structure stiffness matrix.



$$[K]^a + [K]^b + [K]^c + [K]^d = [K]$$

It is usually a banded matrix

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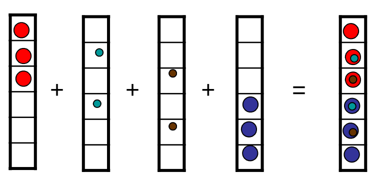
Discrete system: resolution example

6: Expansion to structure dimension of: $\{F\}_p^a$, $\{F\}_{\varepsilon_0}^a$



$$\{F\}_p^a + \{F\}_p^b + \{F\}_p^c + \{F\}_p^d = \{F\}_p$$

7: Assembly of the equilibrium relation of the structure:

$$[K]\{d\} = \{R\} - \{F\}_p - \{F\}_{\varepsilon_0}$$

External concentrated loads and reaction are in $\{R\}$ the distributed ones in $\{F\}_p$ while constraint are seen as knwown components of $\{d\}$. If the problem is elastic, the above expression represents a linear system of equations with {d} as unkwnown.

The Finite Element Method

Discrete system: resolution example

8: Modification of the system due to constraints (deletion of selected rows and columns of the system.

9: Solution of the linear system in 7, through linear algebra techniques. Displacement vector $\{d\}$ is then determined. Afterwards, also constrain reactions may be found.

10: Given $\{d\}$, also the, displacement, stress and strain field $\{\sigma\},\{\varepsilon\}$ within the single elements may be computed.

This is valid for discrete systems only. The procedure can be extended to continuum problems: ***Finite Element Method***.