The Generalized Principle of Inertia Match for Geared Robotic Mechanisms

Dar-Zen Chen  Lung-Wen Tsai
Department of Mechanical Engineering and Systems Research Research Center
University of Maryland, College Park, MD 20742

Abstract

The principle of inertia match has been extended from one-degree-of-freedom (D.O.F.) system to multi-D.O.F. systems. Based on the concept of maximum acceleration capacity, a methodology for the determination of gear ratios in geared robotic mechanisms has been developed. It is found that, at the optimum design, the mass inertia matrix of the input links reflected at the joint-space is equal to that of the major links, and the maximum acceleration capacity is independent of the gear train arrangement. Several two-D.O.F. geared robotic mechanisms have been used as design examples to illustrate the principle. Using this methodology, mechanisms can be designed to yield optimum dynamic performance.

I. Introduction

Various performance measures such as the velocity ellipsoid and the generalized velocity ratio [1, 6], the condition number [7], and the dynamic manipulability index [10] have been proposed for the evaluation of manipulators. Since these performance measures are based on the transformation between the "joint-space" and the "end-effector-space", they are useful for the evaluation and/or design of direct-drive manipulators. However, they are not very helpful for the evaluation and design of manipulators using gear trains or other means for power transmission. For geared robotic mechanisms, the transformation between the "joint-space" and the "actuator-space" must also be considered. Taking this into consideration, Chen and Tsai [4] defined the generalized velocity ratio and acceleration capacity for the design and performance evaluation of geared robotic mechanisms.

For one-D.O.F. geared mechanisms, the principle of inertia match [8] can be used as a guideline for the selection of gear ratios. As for multi-D.O.F. mechanisms, an approach based on kinematic isotropy followed by acceleration capacity optimization was proposed and the concept of two-stage gear-reduction was introduced for the determination of gear ratios by Chen and Tsai [4]. In this paper, a new approach based on the optimization of acceleration capacity alone will be presented. Design equations and optimality conditions will be derived. Several two-D.O.F. robotic mechanisms will be used as numerical examples to demonstrate the principle.

II. Kinematic Equations

In this section, some kinematic equations for geared robotic mechanisms will be briefly reviewed. Figure 1 shows a geared robotic mechanism in conceptual form, where the inputs to the mechanism are the actuators and the output is the end-effector. Let \( \Phi, \Theta, \) and \( X \) be the displacement vectors associated with the actuators, joints, and the end-effector. Then, the joint velocity vector, \( \dot{\Theta} \), and the output velocity vector, \( \dot{X} \), are related by the Jacobian matrix, \( J \), as

\[
\dot{X} = J \dot{\Theta}
\]

Fig. 1: Conceptual diagram of a geared robotic mechanism

And the actuator velocity vector, \( \dot{\Phi} \), is related to the joint velocity vector, \( \dot{\Theta} \), by

\[
\dot{\Phi} = A^T \dot{\Theta}
\]

where \((\cdot)^T\) denotes the transpose of \((\cdot)\). We note that \(A\) is the structure matrix whose elements are functions of gear ratios and each column of \(A\) represents a transmission line in a mechanism [3].

Similarly, the joint torque, \( \tau \), is related to the external force vector \( F \) by

\[
\tau = J^T F
\]

The joint torque, \( \tau \), is related to the actuator torque, \( \xi \), by

\[
\tau = A \xi
\]

III. Dynamic Equations

A. Principle of Inertia Match

Figure 2a shows a one-D.O.F. geared mechanism. The equation of motion is
I, g, denotes the inertia of the input link reflected at the output shaft, Ii the inertia of the input link, IL the inertia of the output link, \( g \) the input torque, \( \xi \) the angular displacement of the output shaft, and \( g = N_2/N_1 \) the gear ratio.

Assume that Ii and IL remain constant regardless of the change in gear ratio and assume that there is no power loss in the gear mesh. Fig. 2b shows the relation between the output shaft acceleration, \( \dot{\xi} \), and the gear ratio, \( g \). It is clear that, given \( \xi, I_i \) and \( I_L \), there exists an optimum gear ratio which yields a maximum output acceleration. At the optimum design, the output acceleration and the gear ratio are given by

\[
\dot{\xi}_{\text{max}} = \frac{\xi}{2 \sqrt{I_i I_L}}
\]

\( g_{\text{opt}} = \frac{I_L}{I_i} \)

Equation (11) provides a torque transformation from the end-effector-space to the actuator-space. Note that both matrices \( A \) and \( M \) are functions of gear ratios.

The question we want to answer is what gear ratios yield the optimum dynamic performance? In eq. (12), \( W_{\xi} \) is a diagonal, positive definite, weighting matrix.

Substituting eq. (11) and its transpose into (12), we obtain

\[
|\xi|^2 = \dot{\xi}^T W_{\xi} \dot{\xi} = 1
\]

Equation (13) represents an acceleration ellipsoid in the end-effector space. As an extension of the principle of inertia match, Chen and Tsai [4] defined the acceleration capacity (A.C.) to be proportional to the volume of the acceleration ellipsoid. They showed that

\[
\text{A.C.} = \frac{\det(J^T W_{\xi} J)}{\det(M)} = \frac{1}{(\det(M))^1/2}
\]

where \( W_{\xi} \) and \( W_{\theta} \) are diagonal, positive definite, weighting matrices.

The problem we want to solve now becomes: what gear ratios yield the optimum acceleration capacity? To answer this question, we will first examine the inertia matrix \( M \), and then seek for the optimum solution.

**IV. The Inertia Matrix M**

It has been shown that there exists an "equivalent open-loop chain" in a geared robotic mechanism [9]. Each link in the equivalent open-loop chain is referred to as a major link while all the other links are called the carried links [5]. As shown in Fig. 3, links 1, 2 and 3 are the major links, and links 4 and 5 are the carried links.

In order to facilitate the dynamic analysis, Chen [5] suggested the following approach. First, all the carried links are treated as being rigidly attached to their carriers and the generalized inertia forces due to the resultant equivalent open-loop linkage are formulated. Second, the effects of relative rotations of the carried links with respect to their carriers are formulated and added to the generalized
inertia forces. Let $M_m$ and $M_r$ be the inertia matrices due to the first and second part of the aforementioned generalized inertia forces, respectively. Then, the inertia matrix $M$ can be written as

$$M = M_m + M_r \quad (15)$$

where both $M_m$ and $M_r$ are positive definite symmetric matrices.

The kinetic energy of carried link $i$ due to relative rotation with respect to its carrier $j$ can be written as

$$K_{ij} = \frac{1}{2} I_i \dot{\theta}_{ij}^2 + \sum_{s=1}^{j-1} (Z_{is} \dot{q}_s) \cdot (Z_{is} \dot{q}_s) \quad (16)$$

where $K_{ij}$ denotes the kinetic energy of link $i$ due to its rotation with respect to link $j$, $I_i$ a unit vector along the "positive" axis of rotation of link $i$, $I_j$ the moment of inertia of link $i$ about its axis of rotation, $\dot{\theta}_{ij}$ rotational speed of link $i$ with respect to link $j$, and $\dot{w}_j$ angular velocity vector of carrier $j$ with respect to the inertia frame.

The angular velocity of a major link $j$, in an open-loop chain, can be written as

$$\dot{w}_j = \sum_{s=1}^{j-1} (Z_{js} \dot{q}_s) \quad (17)$$

where $Z_{qs}$ denotes a unit vector along the $s$-th joint axis in the equivalent open-loop chain, and $\dot{q}_s$ the rate of change of the joint angle $\dot{q}_s$. We note that the unit vectors $Z_{qs}, s = 1, 2, ..., j-1$, are functions of the joint angles.

With the fundamental-circuit equations and the appropriate coaxiality conditions, the rotational speed of carried link $i$ with respect to its carrier $j$ can be written as a linear summation of the joint rates as shown below:

$$\dot{\theta}_{ij} = \sum_{s=j}^{n} (b_{is} \dot{q}_s) \quad (18)$$

where $b_{is}, s = j, j+1, ..., n$, are functions of gear ratios. Furthermore, $b_{is}$ are the elements of the $s$-th column in the structure matrix $A$ defined in [3] if link $i$ is the input link on $r$-th transmission line, and a collection of these $\dot{q}_s$'s forms the actuator velocity vector $\dot{q}$.

Substituting eqs. (17) and (18) into (16), we obtain

$$K_{ij} = I_i \frac{1}{2} \dot{\theta}_{ij}^2 + \sum_{s=j}^{n} (b_{is} \dot{q}_s) \cdot (Z_{is} \dot{q}_s) \cdot (Z_{is} \dot{q}_s) \quad (19)$$

Applying Lagrangian equation on eq. (19) and neglecting the Coriolis and centrifugal terms, we obtain

$$F_i^s = I_i b_{ir} \sum_{s=j}^{n} (b_{is} \dot{q}_s) + \sum_{s=j}^{n} (Z_{is} \dot{q}_s) \cdot (Z_{is} \dot{q}_s) \quad (20a)$$

$$F_i^r = I_i \sum_{s=j}^{n} (b_{is} \dot{q}_s) \cdot (Z_{is} \dot{q}_s) \quad (20b)$$

where $F_i^s$ denotes the generalized inertia force due to the relative motion of a carried link $i$ with respect to its carrier $j$, and associated with $q_s$. Note that the order-of-magnitude for $(Z_{is} \dot{q}_s)$ ranges from -1 to +1, while the $b_{is}$'s are usually one order-of-magnitude larger than $(Z_{is} \dot{q}_s)$. Hence, in general, the first term in eq. (20a) dominates the equation and eq. (20) can be approximated as:

$$F_i^r = \begin{cases} I_i b_{ir} \sum_{s=j}^{n} (b_{is} \dot{q}_s), & \text{for } r \geq j \\ 0, & \text{for } r < j \end{cases} \quad (21)$$

Hence, the contribution of input links to the inertia matrix $M_r$ can be obtained by assembling the coefficients of $\dot{q}_s$ in eq. (21), for all combination of $i$ and $r$, as

$$M_r = I_m A U \hat{A} \quad (22)$$

where

$$I_m = \left( \prod_{i=1}^{n} I_i \right)^{1/n} \quad (23)$$

and $I_i$ is the inertia of $i$-th input link, $U$ is a diagonal scaling matrix with its $(i, i)$ element equal to $I_i / I_m$ and its determinant equal to unity. Note that the contribution to the inertia matrix $M_r$ due to other carried links have been neglected, since they are usually one order-of-magnitude smaller than that due to the input links. It should also be noted that $M_r$ is a function of gear ratios while $M_m$ is a function of the joint angles and the link mass properties. Hence, we can optimize the design of a manipulator only at a predetermined manipulator posture.

V. Acceleration Capacity Optimization

Taking the determinant of eq. (22), yields

$$\det(M_r) = \det(I_m A U A^T) = I_m^n \det(A A^T) \quad (24)$$

From eq. (24), eq. (14) can be further reduced to

$$A.C. = \alpha \lambda \quad (25)$$

where

$$\alpha = \frac{\det(J^T W_s J) \det(W_s)}{\det(M)} \quad (26)$$

$$\lambda = \frac{\det(M_r)^{1/2}}{\det(M)} \quad (27)$$

Note that to maximize acceleration capacity is equivalent to maximize $\lambda$, since $\alpha$ is a constant at a given posture.

A. Two-D.O.F. Systems:

Assume that the structure matrix $A$ takes the following general form:

$$A = \begin{bmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{bmatrix} \quad (28)$$

Then, from eq. (22), inertia matrix $M_r$ can be written as

$$M_r = \begin{bmatrix} \kappa_1 & \kappa_2 \\ \kappa_2 & \kappa_3 \end{bmatrix} \quad (29)$$

where

$$\kappa_1 = I_1 g_{11}^2 + I_2 g_{12}^2 \quad (30)$$
\[ \kappa_2 = I_1 \mathbf{g}_{11} \mathbf{g}_{21} + I_2 \mathbf{g}_{12} \mathbf{g}_{22} \]  
(31)
\[ \kappa_3 = I_1 \mathbf{g}_{21}^2 + I_2 \mathbf{g}_{22}^2 \]  
(32)

Note that the matrix \( \mathbf{M}_r \) contains only three independent parameters, although the number of non-zero elements in the structure matrix can be as many as four. Also note that the structure matrix must have at least three non-zero elements in order for \( \mathbf{M}_r \) to be non-singular and to have non-zero \( \kappa_2 \).

Similarly, the inertia matrix \( \mathbf{M}_m \) can also be expressed in terms of three independent parameters as shown below:

\[ \mathbf{M}_m = \begin{bmatrix} m_1 & m_2 \\ m_2 & m_3 \end{bmatrix} \]  
(33)

Hence, from eq. (15), the inertia matrix \( \mathbf{M} \) is given by

\[ \mathbf{M} = \begin{bmatrix} m + \kappa_1 & m_2 + \kappa_2 \\ m_2 + \kappa_2 & m_3 + \kappa_3 \end{bmatrix} \]  
(34)

Substituting the determinants of eqs. (29) and (34) into eq. (27), we obtain

\[ \lambda = \frac{1}{\sqrt{\left( m + \kappa_1 \right) \left( m_3 + \kappa_3 \right) - \left( m_2 + \kappa_2 \right)^2}} \]  
(35)

Taking the derivative of \( \lambda \) with respect to \( \kappa_i \) for \( i = 1, 2 \) and 3, and equating them to zero, we obtain

\[ (m_3 + \kappa_3) \left( m - \kappa_3 \right) + 2 \kappa_2^2 - \kappa_3 \left( m_2 + \kappa_2 \right)^2 = 0 \]  
(36a)
\[ (m_2 + \kappa_2) \left( m - \kappa_2 \right) + 2 \kappa_1 \kappa_3 - \kappa_2 \left( m_1 + \kappa_1 \right) \left( m_3 + \kappa_3 \right) = 0 \]  
(36b)
\[ (m_1 + \kappa_1) \left( m_3 - \kappa_3 \right) + 2 \kappa_2 - \kappa_1 \left( m_2 + \kappa_2 \right)^2 = 0 \]  
(36c)

Two non-trivial solutions to eqs. (36a)-(36c) are:
\[ \kappa_1 = m_1 \\ \kappa_2 = m_2 \quad \text{and} \quad \kappa_2 = -m_2 \\ \kappa_3 = m_3 \]  
(37)

Since the inertias must be non-negative real numbers, only the former set is a feasible solution. In other word, for two-D.O.F. systems, the optimality condition for maximum acceleration capacity is

\[ \mathbf{M}_r \}_{i,j} = \mathbf{M}_m \}_{i,j} \]  
(38)

provided \( \mathbf{M}_r \) and \( \mathbf{M}_m \) have the same number of independent parameters. Substituting eq. (38) into eq. (15) and the resulting equation into eq. (27), we obtain

\[ \lambda_{opt} = \frac{1}{\sqrt{2 \det(\mathbf{M}_m)}} \]  
(39)

From eqs. (25) and (39), we note that, given \( \mathbf{J} \), \( \mathbf{I}_1 \) and \( \mathbf{I}_2 \), the maximum acceleration capacity of a manipulator at a prescribed posture is independent of the gearing configuration, i.e. the arrangement of transmission lines.

### B. N-D.O.F. Systems:

In the appendix, we have proved that

\[ \frac{\det(\mathbf{M}_r)}{\det(\mathbf{M})} \leq \frac{1}{2^n \det(\mathbf{M}_m)} \]  
(40)

and

\[ \mathbf{M}_r \}_{i,j} = \mathbf{M}_m \}_{i,j} \]  
(41)

is a sufficient condition for the equality sign to hold. This leads to the following theorem.

**Theorem:** For n-D.O.F. geared robotic systems, the acceleration capacity is bounded by the following inequality:

\[ \frac{\det(\mathbf{J}^T \mathbf{W}_k \mathbf{J}) \det(\mathbf{W}_k)}{\det(\mathbf{M}_m)} \leq \left( \frac{1}{2^n \det(\mathbf{M}_m)} \right)^{1/2} \]  
(42)

A sufficient condition for the sign of equality to hold is

\[ \mathbf{M}_r = \mathbf{M}_m \]  
(43)

Equation (43) requires the forms of \( \mathbf{M}_r \) and \( \mathbf{M}_m \) to be compatible. When the equality sign holds, the optimum value of acceleration capacity at a given posture is independent of the gearing configuration.

Equation (43) implies that, at the optimum design, the mass inertia matrix of the input links reflected at the joint-space is equal to that of the major links. We shall call the above theorem the **generalized principle of inertia match** for multi-D.O.F. geared robotic systems.

### VI. Design Examples

For the two-D.O.F. planar manipulators as shown in Figs. 3-4, assume that, at a given posture, the inertia matrix \( \mathbf{M}_m \) takes the form of eq. (34) and the product of Jacobian matrix is:

\[ \mathbf{J}^T \mathbf{W}_k \mathbf{J} = \begin{bmatrix} a & b \\ b & c \end{bmatrix} \]  
(44)

The effect of gearing configuration on the optimum gear ratios is discussed as follows:

#### A. Individual Joint-Drive Manipulator

Figure 3 shows a individual joint-drive manipulator [4]. The structure matrix \( \mathbf{A} \) can be written as

\[ \mathbf{A} = \begin{bmatrix} g_{11} & 0 \\ 0 & g_{22} \end{bmatrix} \]  
(45)

Substituting eq. (45) into eqs. (30)-(32) and the resulting equations into eq. (29), we obtain

\[ \mathbf{M}_r = \begin{bmatrix} \kappa_1 & 0 \\ 0 & \kappa_3 \end{bmatrix} \]  
(46)

where

\[ \kappa_1 = I_1 \mathbf{g}_{11}^2 \]  
(47a)
Since $k_2 = 0$, the forms of $M_1$ and $M_m$ are not compatible with each other and, therefore, eq. (38) cannot be used as a valid solution.

With $k_2 = 0$, eqs. (36a) and (36c) reduce to

\[
\begin{align*}
(m_1 \cdot m_3 - m_2^2) - \kappa_1 \kappa_2 &= 0 \quad (48a) \\
(m_1 \cdot m_3 - m_2^2) + \kappa_1 \kappa_2 &= 0 \quad (48b)
\end{align*}
\]

From eqs. (47a-b) and (48a-b), the optimal conditions for the individual joint-drive manipulator are

\[
\begin{align*}
m_1 \cdot m_3 - m_2^2 &= I_1 g_{11} g_{22} \\
I_1 g_{11}^2 + m_3 &= I_2 g_{22}^2
\end{align*}
\]

Solving eqs. (49a-b) for $g_{11}$ and $g_{22}$ and substituting the results into eq. (43), we have

\[
A = \begin{bmatrix}
\rho_1 (\frac{m_1}{m_3}) \frac{1}{1/4} & 0 \\
0 & \rho_1 (\frac{m_3}{m_2}) \frac{1}{1/4}
\end{bmatrix}
\]

where

\[
\rho_1 = m_1 \cdot m_3 - m_2^2
\]

Assuming $W_0$ is an identity matrix, then from eq. (25), the maximum A.C. can be written as

\[
A.C._{\text{max}} = \frac{(a - b^2)^{1/2}}{2(1 - \frac{1}{I_2}) \left( \frac{1}{I_1} + \frac{1}{m_3} \right)^{1/2}}
\]

B. Gear-Coupled Manipulator

Figure 4 shows a gear-coupled manipulator having three non-zero elements in its structure matrix as shown below:

\[
A = \begin{bmatrix}
g_{11} & g_{12} \\
0 & g_{22}
\end{bmatrix}
\]

where $g_{22} = g_{12}$. Substituting eq. (53) into eq. (39) and solving the resulting equations we obtain

\[
A = \begin{bmatrix}
\rho_1 (\frac{m_1}{m_3}) \frac{1}{1/2} & m_2 \\
0 & \rho_1 (\frac{m_3}{m_2}) \frac{1}{1/2}
\end{bmatrix}
\]

and from eqs. (25), (26) and (39), we obtain

\[
A.C._{\text{max}} = \frac{1}{4} \left( \frac{a - b^2}{1 + I_1 I_2} \right)^{1/2}
\]

Note that a sign change along any column of the structure matrices as shown in eqs. (50) and (54) does not change the optimum acceleration capacity [4].

VII. Numerical Evaluation

For the two-D.O.F. planar manipulators as shown in Figs. 3-4, it can be shown that the Jacobian matrix is given by

\[
J = \begin{bmatrix}
-d_3 S_{12} - d_2 S_1 & -d_3 S_{12} \\
d_3 C_{12} + d_2 C_1 & d_3 C_{12}
\end{bmatrix}
\]

where $d_2 = 22.86$ cm, $d_3 = 17.78$ cm are the lengths of link 2 and link 3, respectively, and where $S_1, C_1, S_{12},$ and $C_{12}$ denote $\sin(\theta_1)$, $\cos(\theta_1)$, $\sin(\theta_1+\theta_2)$, and $\cos(\theta_1+\theta_2)$, respectively. With the end-effector positioned at $[X_1, Y_1] = [22.86, 0]$ as the design reference point, we have (See [4] for detailed derivation)

\[
M_m = \begin{bmatrix}
988 & 29.4 \\
29.4 & 107
\end{bmatrix} \text{ (kg cm$^2$)}
\]

and

\[
J^T W J = \begin{bmatrix}
522.58 & 157.96 \\
157.96 & 316.05
\end{bmatrix}
\]

Assuming $W_x$ and $W_0$ are both identity matrices, we have

\[
J^T W J = \begin{bmatrix}
522.58 & 157.96 \\
157.96 & 316.05
\end{bmatrix}
\]

Let $I_1$ and $I_2$ be 0.088 kg-cm$^2$ and 0.1 kg-cm$^2$, respectively. Then, the optimal gear ratios can be solved for the above two examples. The resulting structure matrices and their maximum acceleration capacities are given in Table 1. It is clear that for the cases in which the forms of $M_1$ and $M_m$ are compatible, the acceleration capacity can always reach a maximum value and is
independent of the gearing configuration.

<table>
<thead>
<tr>
<th>examples</th>
<th>Structure Matrix (A)</th>
<th>A.C.</th>
</tr>
</thead>
</table>
| 1        | \[
\begin{bmatrix}
104.1171 & 0 \\
0 & -32.6417
\end{bmatrix}
\] | 3.12344 |
| 2        | \[
\begin{bmatrix}
103.8970 & 8.9879 \\
0 & 32.7109
\end{bmatrix}
\] | 3.13062 |

Table 1: Structures matrices and acceleration capacities

VIII. Summary

We have extended the principle of inertia match from one-D.O.F. system to multi-D.O.F. systems. A methodology for the determination of optimal gear ratios for geared robotic mechanisms has been developed. The methodology is based on the optimization of acceleration capacity at a given posture. We have shown that individual joint-drive manipulators can be designed to achieve an optimum acceleration capacity, although it cannot be designed to process a kinematically isotropic property[4]. We have also shown that geared-coupled manipulators can be designed to yield a maximum acceleration capacity, provided the forms of Mr and Mm are compatible with each other. At the optimum design, the mass inertia matrix of input links reflected at the joint-space is equal to that of the major links and the maximum acceleration capacity is independent of the gear train arrangement.

Acknowledgement

This work was supported in part by the U.S. Department of Energy under Grant DEFG05-88ER13977 and in part by the NSF Engineering Research Centers Program, NSF D CDR 8803012.

References


Appendix

It has been shown [2] that for positive definite matrices X and Y of order n, the following inequality holds

\[
det(X + Y)^{1/n} \geq \det(X)^{1/n} + \det(Y)^{1/n}
\]  

(A1)

Squaring both sides of eq. (A1), we obtain

\[
det(X + Y)^{2/n} \geq det(X)^{2/n} + \det(Y)^{2/n} + 2 \det(X)\det(Y)^{1/n}
\]  

(A2)

Since it is always true that

\[
det(X)^{2/n} + \det(Y)^{2/n} \geq 2 \det(X)\det(Y)^{2/n}
\]  

(A3)

It follows, from eqs. (A2) and (A3), that

\[
det(X + Y)^{2/n} \geq 2 \det(X)\det(Y)^{2/n}
\]  

(A4)

Taking n/2 power to both sides of eq. (A4), we obtain

\[
det(X + Y)^{1} \geq 2^{n} \det(X)\det(Y)^{1/n}
\]  

(A5)

Dividing eq. (A5) by \([\det(X+Y)\det(Y)]^{1/2}\), yields

\[
[\det(X)]^{1/2} \leq \frac{1}{2^{n/2} \det(Y)]^{1/2}}
\]  

(A6)

Replacing X and Y by Mr and Mm in eq. (A6), respectively, and using eq. (15), we obtain

\[
[\det(M_r)]^{1/2} \leq \frac{1}{2^{n/2} \det(M_m)]^{1/2}}
\]  

(A7)

For Mr = Mm, we have

\[
[\det(M_r + M_m)] = \det(2M_r) = 2^n \det(M_r)
\]  

(A8)

Thus, it can be concluded that M_r = M_m is a sufficient condition for the equality sign in eq. (A7) to hold.